Summation of Series (contd)

1. Sum the following series

$$n\sin\alpha + \frac{n(n-1)}{1\cdot 2}\sin 2\alpha + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\sin 3\alpha$$

$+ \dots$ to n terms.

Soln

Let $S = n\sin\alpha + \dfrac{n(n-1)}{1\cdot 2}\sin 2\alpha + \dfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\sin 3\alpha$

$+ \dots$ to n terms

$$\Rightarrow S = n\sin\alpha + \frac{n(n-1)}{\lfloor 2}\sin 2\alpha + \frac{n(n-1)(n-2)}{\lfloor 3}\sin 3\alpha$$

$+ \dots$ to n-terms.

Let $C = 1 + n\cos\alpha + \dfrac{n(n-1)}{\lfloor 2}\cos 2\alpha + \dfrac{n(n-1)(n-2)}{\lfloor 3}\cos 3\alpha$

$+ \dots$ to $(n+1)$ terms.

$$\therefore C + iS = 1 + n(\cos\alpha + i\sin\alpha) + \frac{n(n-1)}{\lfloor 2}(\cos 2\alpha + i\sin 2\alpha)$$

$$+ \frac{n(n-1)(n-2)}{\lfloor 3}(\cos 3\alpha + i\sin 3\alpha) + \dots \text{ to } (n+1) \text{ terms}$$

$$\Rightarrow C + iS = 1 + n e^{i\alpha} + \frac{n(n-1)}{\lfloor 2} e^{2i\alpha} + \frac{n(n-1)(n-2)}{\lfloor 3} e^{3i\alpha} + \dots$$

to $(n+1)$ terms

$$\Rightarrow C + iS = \left(1 + e^{i\alpha}\right)^n$$

$$\Rightarrow C+iS = \left[1+\cos\alpha+i\sin\alpha\right]^n$$

$$\Rightarrow C+iS = \left[2\cos^2\frac{\alpha}{2}+i\,2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right]^n$$

$$\Rightarrow C+iS = \left[2\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right)\right]^n$$

$$\Rightarrow C+iS = 2^n\cos^n\frac{\alpha}{2}\left(\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right)^n$$

$$\Rightarrow C+iS = 2^n\cos^n\frac{\alpha}{2}\left(\cos\frac{n\alpha}{2}+i\sin\frac{n\alpha}{2}\right)$$

Equating imaginary parts, we get

$$\Rightarrow S = 2^n\cos^n\frac{\alpha}{2}\cdot\sin\frac{n\alpha}{2}$$

Q. Sum the following series

$$\sin\alpha+\frac{1}{2}\sin 2\alpha+\frac{1\cdot3}{2\cdot4}\sin 3\alpha+\dots\text{ to }\infty.$$

Soln. Let $S = \sin\alpha+\frac{1}{2}\sin 2\alpha+\frac{1\cdot3}{2\cdot4}\sin 3\alpha+\dots\text{ to }\infty$

Let $C = \cos\alpha+\frac{1}{2}\cos 2\alpha+\frac{1\cdot3}{2\cdot4}\cos 3\alpha+\dots\text{ to }\infty$

Now, let $S = \sin\alpha+\frac{1}{2}\sin 2\alpha+\frac{1\cdot3}{2\cdot4}\sin 3\alpha+\dots\text{ to }\infty$

$\therefore \; C+iS = (\cos\alpha + i\sin\alpha) + \frac{1}{2}(\cos 2\alpha + i\sin 2\alpha)$

$+ \frac{1.3.}{2.4}(\cos 3\alpha + i\sin 3\alpha) + \cdots \text{ to } \infty$

$\Rightarrow C+iS = e^{i\alpha} + \frac{1}{2}e^{2i\alpha} + \frac{1.3.}{2.4}e^{3i\alpha} + \cdots \text{ to } \infty$

$\Rightarrow C+iS = e^{i\alpha}\left(1 + \frac{1}{2}e^{i\alpha} + \frac{1.3.}{2.4}e^{2i\alpha} + \cdots \text{ to } \infty\right)$

$\Rightarrow C+iS = e^{i\alpha}\left[1 - e^{i\alpha}\right]^{-1/2}$ $\left((1-x)^{-1/2} = 1 + \frac{1}{2}x + \frac{1.3.}{2.4}x^2 + \cdots\right]$

$= \frac{e^{i\alpha}}{\sqrt{1-e^{i\alpha}}} = \frac{e^{i\alpha}}{\sqrt{1-\cos\alpha - i\sin\alpha}}$

$= \frac{e^{i\alpha}}{\sqrt{2\sin^2\frac{\alpha}{2} - 2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}}$

$= \frac{e^{i\alpha}}{\sqrt{2\sin\frac{\alpha}{2}\left(\sin\frac{\alpha}{2} - i\cos\frac{\alpha}{2}\right)}}$

$\Rightarrow C+iS = \frac{\sqrt{2}\,\operatorname{cosec}\frac{\alpha}{2}}{2} \times \frac{e^{i\alpha}}{\sqrt{\cos\left(\frac{\pi}{2}-\frac{\alpha}{2}\right) - i\sin\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)}}$

$$\Rightarrow C+iS = \frac{\sqrt{2\operatorname{cosec}\frac{\alpha}{2}}}{2}\cdot\left(\cos\frac{\pi-\alpha}{2} - i\sin\frac{\pi+\alpha}{2}\right)$$

$$\Rightarrow C+iS = \frac{\sqrt{2\operatorname{cosec}\frac{\alpha}{2}}}{2}\, e^{i\alpha}\left(\cos\frac{\pi-\alpha}{4} + i\sin\frac{\pi-\alpha}{4}\right)$$

$$\Rightarrow C+iS = \frac{\sqrt{2\operatorname{cosec}\frac{\alpha}{2}}}{2}\cdot(\cos\alpha + i\sin\alpha)\left(\cos\frac{\pi-\alpha}{4} + i\sin\frac{\pi-\alpha}{4}\right)$$

$$\Rightarrow C+iS = \frac{\sqrt{2\operatorname{cosec}\frac{\alpha}{2}}}{2}\left[\cos\left(\alpha+\frac{\pi-\alpha}{4}\right) + i\sin\left(\alpha+\frac{\pi-\alpha}{4}\right)\right]$$

$$\Rightarrow C+iS = \frac{\sqrt{2\operatorname{cosec}\frac{\alpha}{2}}}{2}\left[\cos\left(\frac{\pi}{4}+\frac{3\alpha}{4}\right) + i\sin\left(\frac{\pi}{4}+\frac{3\alpha}{4}\right)\right]$$

Equating imaginary parts, we get-

$$S = \frac{\sqrt{2\operatorname{cosec}\frac{\alpha}{2}}}{2}\cdot\sin\left(\frac{\pi}{4}+\frac{3\alpha}{4}\right)$$